

# ZEROES AND RATIONAL POINTS OF ANALYTIC FUNCTIONS

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**ABSTRACT.** For an analytic function  $f(z) = \sum_{k=0}^{\infty} a_k z^k$  on a neighbourhood of a closed disc  $D \subset \mathbb{C}$ , we give assumptions, in terms of the Taylor coefficients  $a_k$  of  $f$ , under which the number of intersection points of the graph  $\Gamma_f$  of  $f|_D$  and algebraic curves of degree  $d$  is polynomially bounded in  $d$ . In particular, we show these assumptions are satisfied for random power series, for some explicit classes of lacunary series, and for solutions of linear differential equations with coefficients in  $\mathbb{Q}[z]$ . As a consequence, for any function  $f$  in these families,  $\Gamma_f$  has less than  $\beta \log^{\alpha} T$  rational points of height at most  $T$ , for some  $\alpha, \beta > 0$ .

## 1. INTRODUCTION

Let  $D_R \subset \mathbb{C}$  be a closed disc, centred at the origin and of radius  $R$ , let  $f : D_R \rightarrow \mathbb{C}$  be an analytic function on a neighbourhood of  $D_R$ , and for any  $d \geq 1$ , let us denote by  $\mathcal{P}_d$  the subspace of polynomials of  $\mathbb{C}[X, Y]$  of degree at most  $d$ . Assuming that for any polynomial  $P \in \mathbb{C}[X, Y]$ ,  $P(z, f(z))$  is not identically zero, or in other words, that  $f$  is transcendental, the quantity

$$Z_d(f) = \sup_{P \in \mathcal{P}_d} \#\{z \in D_R; P(z, f(z)) = 0\},$$

is finite. The integer  $Z_d(f)$  is also the maximum number of intersection points between the graph  $\Gamma_f$  of  $f$  and algebraic curves of degree at most  $d$ . Thus any bound of  $Z_d(f)$  will be called in the sequel a Bézout bound for  $f$ , since such a bound is called a Bézout bound in the case of transverse intersection of two algebraic curves. Moreover in the algebraic case a polynomial bound in the degrees involved holds.

This article studies the growth of  $Z_d(f)$ , as  $d$  goes to infinity; more specifically, we provide certain assumptions on the transcendental analytic function  $f$ , under which  $f$  has, like in the algebraic case, a Bézout bound polynomial in  $d$ .

The asymptotic behavior of  $Z_d(f)$  as  $d$  goes to infinity was studied for different classes of functions, with different tools. In fact,  $Z_d(f) < \infty$  generally holds when  $f$  is definable in an o-minimal structure expanding the real field<sup>1</sup>. But in this very general situation the behaviour of  $f$  with respect to algebraic curves of growing degree is difficult to predict. In the analytic case, considered in the present paper, the following observation is instructive: for any  $\zeta \in ]0, 1[$  there is analytic function  $f$  such

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<sup>1</sup>Note that, on the opposite side, by [26], a polynomial Bézout bound for  $f$  does not imply that  $f$  is polynomial or even definable in some o-minimal structure.

that for a sequence of degrees  $d$  going to infinity,  $Z_d(f) \geq e^{d^c}$  (see, for instance, [34, Example 7.5], [39] or [40], and inequality (5.0.4) of Remark 5.0.9 below). On the other hand, one also knows that for any analytic function  $f$ ,  $Z_d(f)$  is bounded from above by a polynomial in  $d$  of degree 2, for a certain sequence of degrees  $d$  going to infinity (see [22, Theorem 1.1]) and the asymptotic of this upper bound is best possible (see for instance [21, Corollary 2.6])

A polynomial in  $d$  Bézout bound for  $f(z) = e^z$  was obtained in [41]. For entire functions  $f$  of positive order (under some additional conditions) such bounds were obtained in [21, 22]<sup>2</sup> and in [12, 16]. For the Riemann zeta function, and for some other specific functions accurate polynomial bounds were obtained in [11, 30]. Very recently, for solutions of certain types of algebraic differential equations, a polynomial in  $d$  Bézout bound was obtained in [7].

Our approach to the problem of bounding  $Z_d(f)$ , for  $f(z) = \sum_{k=0}^{\infty} a_k z^k$ , is based on a detailed algebraic study of the Taylor coefficients  $a_k$  of  $f$ . It follows a long research line, starting with a classical work of Bautin ([2, 3]). Bautin's discovery was that for analytic families  $f_\lambda(z) = \sum_{k=0}^{\infty} v_k(\lambda) z^k$ , with  $\lambda$  – a parameter ranging in a finite dimensional space, and  $v_k(\lambda)$  – polynomials in  $\lambda$ , the number of zeroes can be bounded in terms of the polynomial ideals  $I_r = \{v_0(\lambda), \dots, v_r(\lambda)\}$  (the Bautin ideals). This approach was further developed in many publications (see, for instance, [15, 17, 24, 37, 43] and references therein). In particular, the case of linear families (that is to say  $v_k(\lambda)$  is linear in  $\lambda$ ) where computations are reduced to linear algebra, was considered in [42]. In the present paper we concentrate on linear families of the form  $P(z, f(z))$ . Notice that some notions introduced below play important role also in Hermite-Padé approximations (see, for instance, [1, 33], and references therein, and Remark 3.2.9 below). We plan to present some instances of this important connection separately.

We will consider polynomials  $P(z, y) = \sum_{0 \leq i, j \leq d} \lambda_{i,j} z^i y^j$ . In this case the parameter vector  $\lambda = (\lambda_{i,j})$  ranges in the space  $\mathbb{C}^m$ , with  $m = (d+1)^2$  and we have

$$f_\lambda(z) = P(z, f(z)) = \sum_{k=0}^{\infty} v_k(\lambda) z^k,$$

with  $v_k(\lambda)$  linear forms in the parameter  $\lambda$ . The specific coefficients of  $v_k(\lambda)$  can be explicitly written through the Taylor coefficients  $a_k$  of  $f$  (see Section 3 below). For each  $k$  we can write

$$\begin{pmatrix} v_0(\lambda) \\ \vdots \\ v_k(\lambda) \end{pmatrix} = M_k \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix},$$

with a  $(k+1) \times m$  matrix  $M_k$ . We define the Bautin index  $b_d = b_d(f)$  as the minimal  $k$  for which the rank of  $M_k$  is equal to  $m$ , and call  $M_{b_d} = M_{b_d}(f)$  the  $d$ -th Bautin matrix of  $f$  (see Definition 2.0.9). The Bautin index  $b_d$  is also the maximal, with respect to  $\lambda$ , multiplicity at 0 of  $f_\lambda(z)$  (see Proposition 2.0.3). Finally, we denote by  $\delta_d = \delta_d(f) > 0$  the maximum of the absolute value of all nonzero minor

<sup>2</sup> For the first time instances of analytic functions (with lacunary Taylor series at the origin) having polynomial Bézout bound with prescribed growth were provided in [22, Corollary 6.7].

determinants of size  $m \times m$  of the Bautin matrix  $M_{b_d}$ . In this way we associate to each transcendental analytic function  $f(z) = \sum_{k=0}^{\infty} a_k z^k$ , two sequences  $(b_d)_{d \geq 1}$ , and  $(\delta_d)_{d \geq 1}$ . For hypertranscendental  $f$  (that is when all the derivatives  $f^{(r)}$  are algebraically independent) we define also a sequence of multiplicities  $(\eta_d)_{d \geq 1}$  and a sequence of non-zero numbers  $(\alpha_d)_{d \geq 1}$  (see Definition 3.3.3). All the four sequences are defined algebraically, through a finite amount (depending on  $d$ ) of the Taylor coefficients  $a_k$  of  $f$ , and they present natural transcendency measures of  $f$ . In the present paper we investigate some basic properties of these sequences, and their role in bounding  $Z_d(f)$ . All the four sequences can be defined for formal power series  $f(z)$  with coefficients in an arbitrary field, although here we work only with complex (or real) analytic functions on  $D_R$ .

The following are our main results:

- (1) We bound  $Z_d(f)$  in terms of  $b_d(f)$  and  $\delta_d(f) > 0$  (Proposition 2.0.11). In particular, for  $b_d(f) \leq R(d)$ , and  $\delta_d(f) \geq e^{-S(d)}$ , with  $R, S$  polynomials in  $d$ , we have  $Z_d(f) \leq T(d)$ , with  $T$  also a polynomial in  $d$ .
- (2) If the Taylor coefficients  $a_k$  of  $f$  are rational, we define  $h_l(f)$  as the maximal denominator of  $a_k$ ,  $k = 0, 1, \dots, l$ . We bound from below  $\delta_d$  in terms of  $b_d$  and  $h_{b_d}$  (Proposition 3.2.7). In particular, for  $b_d(f) \leq R(d)$ , and  $h_l(f) \leq e^{S(l)}$ , with  $R$  and  $S$  polynomials, we show that  $\delta_d(f) \geq e^{-U(d)}$ , and hence  $Z_d(f) \leq T(d)$ , with  $T, U$  also polynomials in  $d$  (Theorem 3.2.8). As an example, we show that this is the case for solution of linear polynomial ODE's with rational coefficients and rational initial values (Theorem 3.5.3). This gives another proof of one of results in [7].
- (3) Consider the Taylor expansion of  $f(z) = \sum_{k=0}^{\infty} a_k(u)(z-u)^k$  at points  $u$  near the origin. Using these expansions we construct, as above, the square matrices  $M_m(u)$ ,  $m = (d+1)^2$ . Assume that the determinants  $\Delta_d(u)$  of  $M_m(u)$  do not vanish identically in  $u$ , and let  $\eta_d(f)$  be the multiplicity of zero of  $\Delta_d(u)$  at  $u = 0$ . We have then

$$\Delta_d(u) = \alpha_d u^{\eta_d} + O(u^{\eta_d+1}), \text{ with } \alpha_d \neq 0.$$

We show that for a certain point  $u \neq 0$  the determinant  $\Delta_d(u)$  can be bounded from below in terms of  $\eta_d$  and  $\alpha_d$ . If the Taylor coefficients  $a_k$  of  $f$  at 0 are rational, we show that  $\alpha_d$  can be bounded from below in terms of  $h_l$ . In particular, for  $\eta_d(f) \leq R(d)$ , and  $h_l(f) \leq e^{S(l)}$ , with  $R$  and  $S$  polynomials, we show that  $\Delta_d(u) \geq e^{-U(d)}$ , and hence  $Z_d(f) \leq T(d)$ , with  $T, U$  also polynomials in  $d$  (Theorem 3.3.6).

- (4) We consider a class of lacunary series  $f$ , similar to the one considered in [20]. We show that for this class the Bautin index  $b_d(f)$  can be explicitly estimated, as well as the Bautin matrices and the determinants  $\delta_d(f)$ . On this base we obtain examples of both polynomial and non-polynomial growth of  $Z_d(f)$  (Theorem 3.4.5).
- (5) Clearly, for a series  $f$  with random Taylor coefficients  $a_k$ , the square Bautin matrices  $M_m(u)$ ,  $m = (d+1)^2$  are non-degenerate. Hence  $b_d(f) = m = (d+1)^2$ . We show that with probability one the determinants  $\Delta_d$  of these

matrices satisfy  $\Delta_d \geq e^{-U(d)}$  (Theorem 4.0.5), and therefore  $Z_d(f) \leq T(d)$ , with  $T, U$  polynomials in  $d$  (Corollary 4.0.6).

- (6) Analytic functions on compact domains with polynomial Bézout bounds have the following remarkable Diophantine property: the number of rational points of height  $\leq T$  in their graph is bounded by a power of  $\log T$ . It means that they have few rational points of given height, since a sharp upper bound for this number, for analytic functions, is in  $C_\epsilon T^\epsilon$ , for any  $\epsilon > 0$ . Therefore our assumptions also provide families of transcendental sets with few rational points, as explained in Section 5 (see Theorem 5.0.8).

The paper is organized as follows. In section 2 we consider general linear families  $f_\lambda$ , we introduce the definitions used throughout the paper, and we give a general bound for  $Z_d(f)$  in terms of the Bautin index and the determinant  $\delta$ . In section 3 we prove the results (1) to (4). In section 4 we prove (5). Finally in Section 5 we give direct Diophantine applications of our analytic Bézout theorems, as mentioned in (6).

## 2. LINEAR FAMILIES OF ANALYTIC FUNCTIONS

Let  $\psi : (\mathbb{C}, 0) \rightarrow (\mathbb{C}^n, 0)$  be an analytic curve, and let

$$(2.0.1) \quad \psi(t) = \sum_{k=1}^{\infty} a_k z^k, \quad a_k \in \mathbb{C}^n,$$

be the Taylor expansion of  $\psi$  at  $0 \in \mathbb{C}$ . We assume that the series (2.0.1) converges in a neighbourhood of the disk  $D_R = \{z \in \mathbb{C}, |z| \leq R\}$ , and that  $\|\psi(t)\| \leq A$  for any  $z \in D_R$ .

For  $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{C}^m$ , let

$$Q_\lambda = \sum_{i=1}^m \lambda_i Q_i,$$

with  $Q_j : (\mathbb{C}^n, 0) \rightarrow \mathbb{C}$ ,  $j = 1, \dots, m$ , analytic functions. We assume that  $Q_j$  are analytic in a neighbourhood of the polydisc of  $\mathbb{C}^n$  of radius  $A$ , and are bounded there by  $B$ .

In what follows we are interested in linear families of analytic functions of the form

$$(2.0.2) \quad f_\lambda(z) = Q_\lambda(\psi(z)) = \sum_{i=1}^m \lambda_i Q_i(\psi(z)).$$

Write

$$(2.0.3) \quad Q_i(\psi(z)) = \sum_{k=0}^{\infty} c_k^i z^k.$$

By our assumptions  $Q_i(\psi(z)) \leq B$  on  $D_R$ ,  $i = 1, \dots, m$ . Hence, by Cauchy's estimates, we have

$$(2.0.4) \quad |c_k^i| \leq \frac{B}{R^k}.$$

For the Taylor development of the linear family  $f_\lambda$  we have

$$(2.0.5) \quad f_\lambda(z) = \sum_{k=0}^{\infty} v_k(\lambda) z^k, \quad v_k(\lambda) = \sum_{i=1}^m c_k^i \lambda_i.$$

Thus the coefficients  $v_k(\lambda)$  of the series (2.0.5) are linear forms in  $\lambda$  and by (2.0.4), for any  $k$

$$(2.0.6) \quad |v_k(\lambda)| \leq \frac{mB|\lambda|}{R^k},$$

where  $|\lambda| = \max\{|\lambda_1|, \dots, |\lambda_m|\}$ .

Now we associate to the family  $f_\lambda$  an integer  $b = b(f_\lambda)$ . Let, for  $i \in \mathbb{N}$ ,  $L_i \subseteq \mathbb{C}^m$  be the linear subspace of  $\mathbb{C}^m$  defined by the equations  $v_0(\lambda) = \dots = v_i(\lambda) = 0$ . We have

$$L_0 \supseteq L_1 \supseteq \dots L_i \supseteq \dots$$

Hence on a certain step  $b$  this sequence stabilizes

$$L_{b-1} \subsetneq L_b = L_{b+1} = \dots = L.$$

Note that  $\lambda \in L_b \iff v_k(\lambda) = 0$ , for  $k = 0, 1, \dots$

**Definition 2.0.1.** We call this number  $b$  the *Bautin index of the family  $f_\lambda$* .

*Remark 2.0.2.* Of course the dimension of the subspaces  $L_i \subset \mathbb{C}^m$ ,  $i \in \mathbb{N}$ , is at most  $m$  and if  $z \mapsto f_\lambda(z)$  is not identically zero for all  $\lambda$ ,  $\dim(L_b) \leq m-1$ . More accurately, if we assume that for  $\lambda \neq 0$  the function  $f_\lambda(t)$  does not vanish identically, then we have  $L_b = \{0\}$ . But the dimension of  $L_i$  can drop at most by one at each step, thus necessarily in this situation  $b \geq m-1$ . This will be the case if  $f(z)$  is a transcendental function for the family of polynomials of degree at most  $d$ , that is for  $\psi(z) = (z, f(z))$  and  $Q_{i,j} = X^i Y^j$  with  $i, j \leq d$ , a classical case mainly considered in the following sections.

On the other hand the Bautin index may be as big as wished. In general one cannot explicitly find the Bautin index of  $f_\lambda$ , since the moments, when the dimension of  $L_i$  drops, are usually difficult to determine.

A first characterization of the Bautin index of the family  $f_\lambda$  is the following.

**Proposition 2.0.3.** *Let us assume that for  $\lambda \neq 0$  the function  $z \mapsto f_\lambda(z)$  is not identically zero, and let us denote by  $\mu$  the maximal multiplicity, with respect to the parameters  $\lambda \neq 0$ , of the Taylor series at the origin of  $f_\lambda(z)$ . Then  $\mu = b$ .*

*Proof.* There exists a parameter  $\lambda$  such that  $f_\lambda(z)$  has multiplicity  $\mu$ , that is such that

$$f_\lambda(z) = v_\mu(\lambda) z^\mu + v_{\mu+1}(\lambda) z^{\mu+1} + \dots,$$

with  $v_\mu(\lambda) \neq 0$ . Therefore  $\lambda \in L_{\mu-1} \setminus L_\mu$ . It follows that  $L_\mu \subsetneq L_{\mu-1}$ , and  $b \geq \mu$ . On the other hand, since no parameter  $\lambda \neq 0$  can cancel  $v_0, \dots, v_\mu$  in the same time,  $L_\mu = \{0\} = L_{\mu+1} = \dots$ , and thus  $b \leq \mu$ .  $\square$

*Remark 2.0.4.* The system  $v_0(\lambda) = \dots = v_{m-2}(\lambda) = 0$ , with  $m$  parameters, always having a nonzero solution, one sees that the maximal multiplicity of  $f_\lambda$  is at least  $m-1$ . Proposition 2.0.3 then implies that  $b \geq m-1$ , that was already noted in Remark 2.0.2.

*Remark 2.0.5.* Under the assumption that for  $\lambda \neq 0$  the function  $z \mapsto f_\lambda(z)$  is not identically zero, and counting zeroes with multiplicity, from Proposition 2.0.3 one observes that

$$\lim_{r \rightarrow 0} \max_{\lambda \neq 0} \# \{z \in D_r, f_\lambda(z) = 0\} \geq b.$$

We give hereafter in Theorem 2.0.7 more accurate relations between the Bautin index and the number of zeroes of the family  $f_\lambda$ , showing in particular that the above inequality is an equality.

Since  $L$  is defined by  $v_0(\lambda) = \dots = v_b(\lambda) = 0$ , any linear form  $\ell(\lambda)$ , which vanishes on  $L$ , can be expressed as a linear combination of  $v_0, \dots, v_b$ . Now a basis  $(v_{i_1}, \dots, v_{i_\sigma})$ ,  $i_1, \dots, i_\sigma \in \{0, \dots, b\}$ , of the space of linear forms vanishing on  $L$  being chosen among the elements of the family  $(v_0, \dots, v_b)$ , there exists a constant  $\tilde{c} > 0$ , depending on this basis, such that for any  $\ell$  with  $\ell|_L \equiv 0$ , we have

$$(2.0.7) \quad \ell(\lambda) = \sum_{j=1}^{\sigma} \mu_j v_{i_j}(\lambda), \quad \mu_j \in \mathbb{C}, \text{ and } |\mu_j| \leq \tilde{c} \|\ell\|, j = 1, \dots, \sigma.$$

where for  $\ell(\lambda) = \sum_{i=1}^m \alpha_i \lambda_i$ ,  $\|\ell\| = \max_i |\alpha_i|$ .

*Notation 2.0.6.* We denote by  $c = c(f_\lambda) > 0$  the minimum of the constants  $\tilde{c}$  satisfying (2.0.7).

An effective estimation of  $c$  is difficult, in general. However, if the Bautin index  $b(f_\lambda)$  is known,  $c(f_\lambda)$  can be estimated via a finite computation in terms of the Taylor coefficients of  $\psi$  and of  $Q_i$  (see Proposition 2.0.11 below).

Now let a family  $f_\lambda$  be given, and let the Bautin index  $b(f_\lambda)$  and the constant  $c(f_\lambda)$  of this family be defined as above.

**Theorem 2.0.7.** *Zeroes being counted with multiplicity, we have the following uniform bounds with respect to the parameter  $\lambda$ , when  $f_\lambda \not\equiv 0$ .*

(1) *The maximal number of zeroes of  $f_\lambda$  in the disk  $D_{\frac{R}{4}}$  is at most*

$$5b \log \left( 4 + 2c(b+1) \frac{B}{R^b} \right) \quad \text{if } R \leq 1,$$

$$\text{and } 5b \log (4 + 2c(b+1)B) \quad \text{if } R \geq 1,$$

(2) *This maximal number of zeroes of  $f_\lambda$  is at most  $b$  in  $D_\rho$ , where*

$$\rho = \frac{R}{e^{10b+2} \max(2, c(b+1)B \max(\frac{1}{R}, 1)^b)}.$$

*Proof.* For any  $\lambda \in \mathbb{C}^m$ , and for any  $j \geq b+1$ , by the definition of  $c$  and from the bound (2.0.4) (see Theorem 1.1 of [42] and the last inequality in its proof), we have

$$(2.0.8) \quad |v_j(\lambda)| R^j \leq c(b+1)B \max\left(\frac{1}{R}, 1\right)^b \max_{i=0, \dots, b} |v_i(\lambda)| R^i.$$

Then, the bounds to prove on the number of zeroes are consequences of (2.0.8) and [37].  $\square$



*Remark 2.0.8.* Note that under the assumption that for  $\lambda \neq 0$  the function  $z \mapsto f_\lambda(z)$  is not identically zero, and counting zeroes with multiplicity, from Theorem 2.0.7 and Remark 2.0.5 one has

$$\lim_{r \rightarrow 0} \max_{\lambda \neq 0} \#\{z \in D_r, f_\lambda(z) = 0\} = b.$$

This infinitesimal maximal number of zeroes is called the *cyclicity of  $Q_\lambda$  on  $\psi$*  (see [42]).

In what follows we develop explicit bounds on the number of zeroes of  $f_\lambda$  in  $D_{\frac{R}{4}}$  in terms of the coefficients  $c_k^i$ , with  $i = 1, \dots, m$ ,  $k = 0, \dots, b$ , i.e., ultimately, in terms of the Taylor coefficients of  $\psi$  and of  $Q_i$  up to the order  $b$ . By Theorem 2.0.7, this amounts bounding the constant  $c(f_\lambda)$  introduced in Notation 2.0.6. Note also that such a bound is given by a bound on the coefficients  $\mu_j, \dots, \mu_\sigma$  of the system (2.0.7), since in this system one can consider only linear forms  $\ell$  of norm 1. Let us denote the dimension of the stabilized subspace  $L = L_b$  by  $s = m - \sigma$  ( $\leq m - 1$ ). All the information we need, as we will see, is encoded in the rank  $\sigma$  matrix  $M = (c_k^i)$ ,  $k = 0, \dots, b$ ,  $i = 1, \dots, m$ , with  $b + 1$  lines and  $m$  columns. With our notation,  $M = M_b$  is defined by

$$\begin{pmatrix} v_0(\lambda) \\ \vdots \\ v_b(\lambda) \end{pmatrix} = M_b \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix}.$$

**Definition 2.0.9.** With the above notation, the  $(b + 1) \times m$  matrix  $M = M_b$  is called the *Bautin matrix of the family  $f_\lambda$* .

*Notation 2.0.10.* We will denote by  $\delta > 0$  the maximum of the absolute value of all nonzero minor determinants of size  $\sigma \times \sigma$  of the Bautin matrix  $M$ .

**Proposition 2.0.11.** *Let  $f_\lambda$  be given as above. Then*

$$c(f_\lambda) \leq \sigma \frac{(B\sqrt{\sigma})^{\sigma-1}}{\delta R^{\beta(\sigma-1)}}, \quad \text{where } \beta = b \text{ if } R \leq 1 \text{ and } \beta = \frac{\sigma}{2} \text{ if } R \geq 1.$$

*In turn, in the disk  $D_{\frac{R}{4}}$ , the maximal number  $Z(f_\lambda)$  of zeroes of the family  $f_\lambda$ , with respect to the parameter  $\lambda$ , satisfies*

$$\begin{aligned} Z(f_\lambda) &\leq 5b \log(4 + 2(b + 1)\sqrt{\sigma} \frac{(B\sqrt{\sigma})^\sigma}{\delta R^{b\sigma}}) \quad \text{if } R \leq 1, \\ \text{and } Z(f_\lambda) &\leq 5b \log(4 + 2(b + 1)\sqrt{\sigma} \frac{(B\sqrt{\sigma})^\sigma}{\delta R^{\frac{\sigma}{2}(\sigma-1)}}) \quad \text{if } R \geq 1, \end{aligned}$$

*Proof.* Let  $\hat{M}$  be a submatrix of  $M$  of size  $\sigma \times \sigma$ , with the absolute value of the determinant equal to  $\delta$ , according to notation 2.0.10.  $\hat{M}$  is obtained from  $M$  by the choice of the lines  $i_1 < \dots < i_\sigma$  of  $M$  (corresponding to the choice  $(v_{i_1}, \dots, v_{i_\sigma})$  for a basis of the space of linear forms cancelling on  $L$ ), and the choice of certain  $\sigma$  columns, say, the first  $\sigma$  columns in  $M$ . When  $\|\ell\| \leq 1$ , (2.0.7) gives the linear system

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_\sigma \end{pmatrix} = {}^t \hat{M} \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_\sigma \end{pmatrix}$$

with  $|\alpha_j| \leq 1$ ,  $j = 1, \dots, \sigma$ . Therefore, by the Cramer rule, each  $\mu_j$  satisfies

$$(2.0.9) \quad |\mu_j| \leq \frac{\sigma \hat{\delta}}{\delta},$$

where  $\hat{\delta}$  is the maximum of the absolute values of  $(\sigma - 1) \times (\sigma - 1)$  sub-minors of  $\hat{M}$ . Next, by (2.0.4), we have

$$|c_{i_j}^i| \leq \frac{B}{R^{i_j}} \leq \frac{B}{R^b} \text{ if } R \leq 1$$

$$\text{and } |c_{i_j}^i| \leq \frac{B}{R^{i_j}} \leq \frac{B}{R^j} \text{ if } R \geq 1.$$

Consequently, the length of the  $j$ -th row-vectors in  $(\sigma - 1) \times (\sigma - 1)$  sub-minors of  $\hat{M}$  does not exceed

$$\frac{B\sqrt{\sigma-1}}{R^b} \text{ if } R \leq 1, \text{ and } \frac{B\sqrt{\sigma-1}}{R^j} \text{ if } R \geq 1$$

Interpreting the determinant as the volume of the span of its row-vectors, we conclude that

$$\sigma \hat{\delta} \leq \sigma \frac{(B\sqrt{\sigma-1})^{\sigma-1}}{R^{b(\sigma-1)}} \leq \sigma \frac{(B\sqrt{\sigma})^{\sigma-1}}{R^{b(\sigma-1)}} \text{ if } R \leq 1$$

$$\text{and } \sigma \hat{\delta} \leq \sigma \frac{(B\sqrt{\sigma-1})^{\sigma-1}}{R^{\frac{\sigma}{2}(\sigma-1)}} \leq \sigma \frac{(B\sqrt{\sigma})^{\sigma-1}}{R^{\frac{\sigma}{2}(\sigma-1)}} \text{ if } R \geq 1.$$

This bound, combined with (2.0.9) and Theorem 2.0.7, completes the proof of Proposition 2.0.11.  $\square$

*Remark 2.0.12.* From the beginning of this section, we have assumed that the analytic map  $\psi$  is defined on a disc centred at the origin and that  $\psi(0) = 0$ . This choice is harmless since, in case  $\psi$  is defined on a ball centred at  $a$ , one can consider  $\phi(z) = \psi(z + a) - \psi(a)$  and the bounds given in Theorem 2.0.7 and Proposition 2.0.11 for  $\phi$  and the family  $Q_i(w + \psi(a))$  are the same bounds for  $\psi$  and the family  $Q_i(w)$  when Taylor series are considered at  $a$  instead of 0.

*Remark 2.0.13.* A classical application of bounds given in Proposition 2.0.11 is for analytic plane curves  $\psi(z) = (z, f(z))$  and the family  $Q_i(X, Y)$  of two variables monomials of total degree at most some integer  $d$ . In this case we consider that  $f$  is given by its Taylor series at the origin and that this series converges on the disc  $D_R$  of radius  $R$  and centred at the origin, by Remark 2.0.12. The curve  $\psi(z)$  is the standard parametrization of the graph of  $f$ . In this setting we will provide bounds for the number of zeroes of  $P_d(z, f(z))$  on  $D_R$  that are uniform with respect to coefficients of polynomials  $P_d$  of degree at most  $d$ . Now note that in this situation one can only consider functions  $f$  that are bounded by 1 on  $D_R$ , since a uniform bound on the number of zeroes of  $P_d(z, \frac{1}{N}f(z))$ , where  $N$  bounds  $f(z)$  on  $D_R$ , provides a uniform bound on the number of zeroes of  $P_d(z, f(z))$ . In the same way one can as well assume for simplicity that  $f$  is analytic on the unit disc, up to applying the bounds provided by Proposition 2.0.11 to the new function  $g(z) = f(Rz)$ , since a uniform bound on the zeroes of  $P_d(w, g(w))$  on  $D_1$  is a uniform bound on the zeroes of  $P_d(z, f(z))$  on  $D_R$ . Of



course the same rescaling effects apply in the same way for the family  $X^i Y^j, i, j \leq d$ , of two variables monomials. Nevertheless those reductions are not always possible for any family of analytic functions  $Q_j$  and for any analytic curve  $\psi$ .

Finally when one restricts to analytic functions bounded by 1 on  $D_1$  and for the family  $Q_{i,j} = X^i Y^j, i + j \leq d$  or for the family  $X^i Y^j, i, j \leq d$  of two variables monomials, one can take 1 for  $B$  as bound for  $Q_{i,j}(z, f(z))$  on  $D_1$ . Note that in this case, from Proposition 2.0.11 we deduce the following statement.

**Corollary 2.0.14.** *Let  $f_\lambda$  be given as above with  $R = 1$  and  $B = 1$ , then in the disk  $D_{\frac{1}{4}}$  the maximal number  $Z(f_\lambda)$  of zeroes of the family  $f_\lambda$ , with respect to the parameters  $\lambda$ , satisfies*

$$Z(f_\lambda) \leq 5b \log(4 + 2(b + 1) \frac{e^{\sigma \log \sigma}}{\delta}).$$

*Proof.* After taking  $B = 1$  and  $R = 1$  in the last inequality of Proposition 2.0.11, observe that  $\sqrt{\sigma}^{\sigma+1} = e^{(\sigma+1)\frac{1}{2} \log \sigma} \leq e^{\sigma \log \sigma}$ .  $\square$

### 3. BÉZOUT BOUNDS FOR TRANSCENDENTAL ANALYTIC CURVES

**3.1. Families of polynomials.** From now on we consider the classical case of the family of polynomials with degree at most a fixed integer  $d$ . That is to say, for a given analytic function  $f : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$  we want to bound the number of zeroes of  $P_d(z, f(z))$ , for  $P_d$  a two variables polynomial of degree at most  $d$ . Considering Remarks 2.0.12 and 2.0.13, for simplicity we assume that  $f$  sends 0 to 0, is given by its converging Taylor series on the unit disc  $D_1$ , and is bounded by 1 on  $D_1$ .

In fact, we consider the problem in the following (essentially equivalent) form <sup>3</sup>: give a bound for

$$\mathcal{Z}_d(f) := \max_{p_j \in \mathbb{C}[z], \deg p_j \leq d} \# \{z \in D, \sum_{j=0}^d p_j(z) f^j(z) = 0\}$$

for some disc  $D \subset D_1$  centred at the origin. Denoting  $p_j(z) = \sum_{i=0}^d \lambda_{j,i} z^i$ , the sum

$$\sum_{j=0}^d p_j(z) f^j(z) \text{ has the form } \sum_{0 \leq i, j \leq d} \lambda_{j,i} z^i f^j(z).$$

With the notation of Section 2, we have to consider the family of monomials  $Q_{i,j} = X^i Y^j, i, j \leq d$ , and the analytic function  $\psi(z) = (z, f(z))$ , therefore here the number of parameters is  $m = (d + 1)^2$ . In order to guarantee  $\mathcal{Z}_d(f) < \infty$  for any  $d$  as soon

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<sup>3</sup> The family  $\sum_{0 \leq i, j \leq d} \lambda_{j,i} z^i f^j(z)$  considered here presents the advantage to have a slightly more

symetric Bautin matrix then the family  $\sum_{i+j \leq d} \lambda_{j,i} z^i f^j(z)$ . Of course one can easily deduce the Bautin

matrix of the second family from the Bautin matrix of the first family, and the problem could be considered only for the family  $\sum_{i+j \leq d} \lambda_{j,i} z^i f^j(z)$  as well. In particular note that  $Z_d(f) \leq \mathcal{Z}_d(f) \leq$

$\mathcal{Z}_{2d}(f)$ , and therefore a polynomial bound for the maximal number of zeroes of one family gives rise to a polynomial bound for the maximal number of zeroes of the other family.

as some  $p_j$  is not zero, we assume that  $f$  is transcendental, that is no polynomial restricted to the graph of  $f$  vanishes identically. By Remark 2.0.2, the Bautin index  $b = b_d$  of this family satisfies  $b \geq m - 1 = d^2 + 2d$  and the corresponding Bautin matrix  $M = M_b$  has at least  $m = (d + 1)^2$  lines (and exactly  $m$  columns). Furthermore the dimension  $s$  of the space  $L_b$  is 0 and therefore the dimension  $\sigma$  of the space generated by  $v_0, \dots, v_b$  is  $m$ .

*Notation 3.1.1.* For  $i, j \in \mathbb{N}$ , we denote by  $a_i^j$  the  $i$ th Taylor coefficient at the origin of the  $j$ th power  $f^j$  of  $f$ . Namely,  $a_i^j = \frac{1}{i!}(f^j)^{(i)}(0)$ .

With this notation, a direct computation shows that

$$M = \begin{pmatrix} 1 & 0 & \text{---} & 0 & a_0^1 & 0 & \text{---} & 0 & \cdots & a_0^d & 0 & \text{---} & 0 \\ \left| \right. & & & \left| \right. & \left| \right. & & & \left| \right. & & \left| \right. & & & \left| \right. \\ 0 & & & 0 & a_{d-1}^1 & a_{d-2}^1 & & 0 & \cdots & a_{d-1}^d & & & 0 \\ 0 & & & 1 & a_d^1 & a_{d-1}^1 & & a_0^1 & \cdots & a_d^d & & & a_0^d \\ 0 & & & 0 & a_{d+1}^1 & a_d^1 & & a_1^1 & \cdots & a_{d+1}^d & & & a_1^d \\ \left| \right. & & & \left| \right. & \left| \right. & & & \left| \right. & & \left| \right. & & & \left| \right. \\ 0 & 0 & \text{---} & 0 & a_b^1 & a_{b-1}^1 & \text{---} & a_{b-d}^1 & \cdots & a_b^d & a_{b-1}^d & \text{---} & a_{b-d}^d \end{pmatrix}$$

By Corollary 2.0.14 any non zero minor determinant of size  $m \times m$  of  $M$  will provide a bound for  $\mathcal{Z}_d(f)$  on  $D_{\frac{1}{4}}$ , since in Corollary 2.0.14,  $\delta$  is the maximum of all non zero minor determinants of size  $m \times m$  of  $M$ . But such a minor has to contain the first  $d + 1$  lines of  $M$ , as well as the last line. One sees that the absolute value  $\Delta$  of any non zero determinant of  $(d^2 + d) \times (d^2 + d)$  minor of the following matrix

$$(3.1.1) \quad \tilde{M} = \begin{pmatrix} a_{d+1}^1 & \text{---} & a_1^1 & \cdots & a_{d+1}^d & \text{---} & a_1^d \\ \left| \right. & & \left| \right. & & \left| \right. & & \left| \right. \\ a_b^1 & \text{---} & a_{b-d}^1 & \cdots & a_b^d & \text{---} & a_{b-d}^d \end{pmatrix}$$

provides on  $D_{\frac{1}{4}}$  the Bézout bound

$$(3.1.2) \quad \mathcal{Z}_d(f) \leq 5b \log(4 + 2(b + 1)) \frac{e^{2(d+1)^2 \log(d+1)}}{\Delta}.$$

**3.2. Bézout bound through the transcendency index.** We start this subsection by the following definition of a notion of measure of the local transcendency of an analytic transcendental function.

**Definition 3.2.1.** For a transcendental analytic function  $f : D_1 \rightarrow \mathbb{C}$  and for any  $d \geq 1$  the  $d$ -th transcendency index  $\nu_d$  of  $f$  is the maximal (with respect to all nonzero

polynomials  $P_d \in \mathcal{P}_d$ ) multiplicity at 0 of the function  $g(z) = P_d(z, f(z))$ . The non-decreasing sequence  $\nu(f) = (\nu_1, \dots, \nu_d, \dots)$  is called the *transcendence sequence* of  $f$ .

*Remark 3.2.2.* The  $d$ -th transcendence index measures the maximal order of contact at the origin between the graph of  $f$  and algebraic curves of degree at most  $d$ . The higher this index is, the less  $f$  seems transcendental, since infinite  $\nu_d$  means that  $f$  is algebraic.

*Remark 3.2.3.* As already observed in Proposition 2.0.3,  $\nu_d$  is the Bautin index of the linear family associated to  $f$  and the monomials  $X^i Y^j$  of degree  $i + j$  at most  $d$ . Since the number of monomials in two variables of degree at most  $d$  is  $(d + 1)(d + 2)/2$ , by Remark 2.0.2,  $\nu_d \geq (d^2 + 3d)/2$  always.

*Remark 3.2.4.* Using the notation  $b$ , as in the beginning of section 3, for the Bautin index of the family  $\sum_{0 \leq j \leq d} p_j(z) f^j(z)$ , with  $\deg p_j \leq d$ , one has by Proposition 2.0.3 that  $b = \mu$ , where  $\mu$  is the maximal multiplicity, with respect to the coefficients of the polynomials  $p_j$ , of this family. Therefore one gets  $\nu_d \leq b = \mu \leq \nu_{2d}$ . In particular, inequality (3.1.2) gives the following proposition.

**Proposition 3.2.5.** *Let  $f : D_1 \rightarrow \mathbb{C}$  be a transcendental analytic function with transcendence sequence  $(\nu_d)_{d \geq 1}$ , then on  $D_{\frac{1}{4}}$*

$$\mathcal{Z}_d(f) \leq 5\nu_{2d} \log(4 + 2(\nu_{2d} + 1) \frac{e^{2(d+1)^2 \log(d+1)}}{\Delta}),$$

where  $\Delta$  is the absolute value of a certain non zero minor determinant of size  $(d^2 + d) \times (d^2 + d)$  of the matrix  $\tilde{M}$  defined at (3.1.1).

Now we assume that  $f(z) = \sum_{k=0}^{\infty} a_k z^k$  has rational Taylor coefficients  $a_k = \frac{m_k}{p_k}$ , with the greatest common divisor  $m_k \wedge p_k$  of  $m_k, p_k$  equal to 1 and  $p_k > 0$ , and let us introduce the following notation.

*Notation 3.2.6.* For any  $l \geq 1$ , let  $h_l = \max\{p_k; k = 1, \dots, l\}$ .

Under this assumption we can bound from below the non zero determinant of the  $(d^2 + d) \times (d^2 + d)$  minors of  $\tilde{M}$ , in terms of the transcendence index  $\nu_{2d}$  and the height bound  $h_{\nu_{2d}}$ .

**Proposition 3.2.7.** *For  $f$  as above and for  $\Delta$  the absolute value of a non zero minor determinant of size  $(d^2 + d) \times (d^2 + d)$  of  $\tilde{M}$  we have*

$$\Delta \geq h_{\nu_{2d}}^{-d^2(d+1)\nu_{2d}}.$$

*Proof.* For simplicity write  $h$  for  $h_{\nu_{2d}}$  and  $\nu$  for  $\nu_{2d}$ . Using the notation of 3.1.1, write the coefficients  $a_k^1$ ,  $k \leq \nu$ , as rational numbers having the same denominator  $D$ , this common denominator being at most  $h^\nu$ . Next we write the coefficients  $a_i^j$ , for  $1 \leq j \leq d$ ,  $1 \leq i \leq \nu$ , as rational numbers having the same denominator  $D^d$ . Indeed,  $a_i^j$  are sums of the products  $a_{i_1}^1 \cdots a_{i_j}^1$ ,  $i_1 + \cdots + i_j = i$ , with  $j \leq d$ . Therefore the

determinant of a  $(d^2 + d) \times (d^2 + d)$  minor of  $\tilde{M}$  can be written as a rational number having for denominator

$$D^{d^2(d+1)} \leq h^{d^2(d+1)\nu}.$$

But then such a non zero determinant cannot be smaller in absolute value than  $h^{-d^2(d+1)\nu}$ .  $\square$

An important special case is when there exist polynomials  $R(d)$  and  $S(d)$ , with positive coefficients, such that

$$(3.2.1) \quad \nu_d \leq R(d), \quad h_l \leq e^{S(l)}, \quad d, l \geq 1$$

Under this condition we can guarantee that  $\mathcal{Z}_d(f)$  grows at most polynomially in  $d$ .

**Theorem 3.2.8.** *Assume that  $f$  has rational Taylor coefficients at the origin, and that the growth conditions (3.2.1) are satisfied. Then on  $D_{\frac{1}{4}}$*

$$\mathcal{Z}_d(f) \leq T(d),$$

for a certain polynomial  $T$ .

*Proof.* Under condition (3.2.1) and by Proposition 3.2.7, since  $S$  is an increasing function, we have

$$\frac{1}{\Delta} \leq h_{\nu_{2d}}^{d^2(d+1)\nu_{2d}} \leq e^{S(R(2d))d^2(d+1)R(2d)} = e^{U(d)}.$$

Now by Proposition 3.2.5, on  $D_{\frac{1}{4}}$ , we easily have for instance

$$\begin{aligned} \mathcal{Z}_d(f) &\leq 5R(2d) \log(4 + 2(R(2d) + 1)) e^{2(d+1)^2 \log(d+1)} e^{U(d)} \\ &\leq 5R(2d) \log(4R(2d)) e^{2(d+1)^3} e^{U(d)} \\ &\leq 10R^2(2d) + 10R(2d)(2(d+1)^3 + U(d)) \end{aligned}$$

$\square$

*Remark 3.2.9.* Producing instances of Taylor series  $f(z) = \sum_{k \geq 0} a_k z^k$  converging on  $D_1$ ,

with rational coefficients  $a_k$  having denominators bounded from above by  $e^{S(k)}$ , where  $S$  is a certain polynomial, is easy. Nevertheless the second assumption of Theorem 3.2.8, concerning the growth of the transcendence sequence of  $f$ , is more difficult to control. A polynomial bound  $\nu_d(f) \leq R(d)$  is known for solutions of some classes of algebraic ODE's (see [6, 25, 32, 31]). We expect such a bound to hold for Taylor series produced by some natural classes of recurrence relations. In Section 3.4 we give conditions on lacunarity of the series  $f$  that allow estimates of the growth of  $\nu_d(f)$ . In general, we consider bounding of the growth of the transcendence sequence of  $f$  as an important open question.

If we consider polynomials  $P(z, y) = p_1(z)y + p_0(z)$  of degree 1 in  $y$ , with  $p_0(z), p_1(z)$  of degree  $d$  in  $z$ , we are in the framework of the classical Padé approximation. In this case the sequence of maximal multiplicities  $\mu_d = \mu_d(f)$  of  $g(z) = P(z, f(z)) = p_1(z)f(z) + p_0(z)$  has the following remarkable description (see, for instance, [33]): let

$$f(z) = \frac{1}{q_1(z) + \frac{1}{q_2(z) + \dots}}, \quad \deg q_l = s_l, \quad l = 1, 2, \dots,$$

be a continued fraction representation of the series  $f(z)$ . Then  $\mu_d = s_1 + s_2 + \dots + s_d$ .

For polynomials  $P(z, y) = p_0(z)$  of degree 0 in  $y$  the behavior of  $\mu_d$  was studied in [24], in particular, it was related there to linear non-autonomous recurrence relations for the Taylor coefficients of  $f$ .

**3.3. Bézout bound through the Bautin multiplicity sequence.** In the previous section 3.2, in Proposition 3.2.7, some minors of the matrix  $\tilde{M}$  were bounded from below, in terms of the transcendence index  $\nu_{2d}$  and the height bound  $h_{\nu_{2d}}$ . On this base, on  $D_{\frac{1}{4}}$ ,  $\mathcal{Z}_d(f)$  was bounded from above by a polynomial in  $d$  (see Theorem 3.2.8), under the condition that the sequences  $(\nu_d)_{d \geq 1}$  and  $(\log h_d)_{d \geq 1}$  are polynomially bounded.

A special case in which the transcendence index (or thanks to the double inequality of Remark 3.2.4, in which the Bautin index  $b$  itself of the family  $\sum_{j=0}^d p_j f^j, \deg p_j \leq d$ ) is polynomially bounded is the case that  $b$  is minimal, that is equal to  $d^2 + 2d$ . In this case the matrix  $\tilde{M}$  of (3.1.1) is an invertible square matrix of size  $d^2 + d$  with the same determinant as the Bautin matrix  $M$ .

*Notation 3.3.1.* Being zero or not, let us call this  $(d^2 + d) \times (d^2 + d)$  determinant *the Bautin determinant* of the family  $\sum_{j=0}^d p_j f^j, \deg p_j \leq d$ , and let us denote it by  $\Delta_d$ .

So far the study has been done by looking at Taylor series at the origin. Let us now allow Taylor expansions of  $f$  at points  $z$  near the origin. In this situation, the Bautin matrix  $M$ , as well as its submatrix  $\tilde{M}$ , have entries  $a_i^j(z)$  that are analytic functions in the variable  $z$ . To emphasize this dependency, we adopt the notation  $\tilde{M}(z)$  and  $\Delta_d(z)$  (and keep the notation  $\tilde{M}$  and  $\Delta_d$  for  $z = 0$ ). We shall assume that for each degree  $d$  the Bautin determinant  $\Delta_d(z)$ , as a function of  $z$ , does not vanish identically. This is in particular true when  $f(z)$  is a hypertranscendental function, i.e. all the derivatives are algebraically independent. In this situation, for a generic base point  $z$ ,  $\Delta_d(z) \neq 0$ ,  $\nu_d(z) \leq d^2 + 2d$ , and therefore the transcendence index  $\nu_d(z)$  is polynomially bounded. The study of section 3.2 could then be done by shifting the origin at some generic point  $z$ , however in this translation one loses the control on the rationality of the coefficients of the Taylor expansion of  $f$ , an assumption that is necessary to formulate at some fixed point (namely the origin, for simplicity), since this assumption makes no sense at generic points  $z$ . Nevertheless, still in the case that  $\Delta_d(z)$  does not vanish identically, one can use the following dichotomy:

- when  $\Delta_d(0) \neq 0$ , as just observed, we are in particular in the frame of Section 3.2 where  $\nu_d$  is polynomially bounded,
- when  $\Delta_d(0) = 0$ , we can expand  $\Delta_d(z)$  as a non zero Taylor series in  $z$  at the origin and study the multiplicity of this expansion with respect to  $d$ . We will see in Theorem 3.3.6 that when the sequence of these multiplicities has at most a polynomial growth, then a Bézout bound for  $f$  is still possible.

*Remark 3.3.2.* This dichotomy means that when some transversality defect for  $f$  is quantitatively well controlled (through the multiplicity of  $\Delta_d(z)$ ), then a good zero-counting bound is possible, and for instance, in turn, a good bound for the density of rational points of bounded height in the graph of  $f$  will also be possible (see Theorem 5.0.8).

**Definition 3.3.3.** For any  $d \geq 1$ , the  $d$ -th Bautin multiplicity  $\eta_d$  of  $f$  is the multiplicity at 0 of the Bautin determinant  $\Delta_d(z)$ , considered as an analytic function of  $z$ . The sequence  $\eta(f) = (\eta_d)_{d \geq 1}$  is called the *Bautin multiplicity sequence* of  $f$ .

Consequently, for each  $d \geq 1$ , we can write

$$(3.3.1) \quad \Delta_d(z) = \alpha_d z^{\eta_d} + O(z^{\eta_d+1}), \text{ with } \alpha_d \neq 0.$$

We assume in what follows that the function  $f$  is analytic on the disc, say,  $D_8$ , and is bounded there by 1.

The goal of this section is to show that one can explicitly bound from below the Bautin determinants  $\Delta_d(z)$  at points  $z$  near the origin, in terms of the sequence  $(\eta_d)_{d \geq 1}$  and under the assumption that the coefficients of the Taylor expansion on  $f$  at the origin are rational numbers with controlled denominators. This will be sufficient to obtain a Bézout bound for  $f$  in a disc centred at  $z$  (Theorem 3.3.6).

In this aim, let us first prove that one can bound from below the Bautin determinants  $\Delta_d(z)$ , at points  $z$  near the origin, in terms of the sequences  $(\eta_d)_{d \geq 1}$  and  $(\alpha_d)_{d \geq 1}$ .

**Lemma 3.3.4.** For any  $d \geq 1$  and  $|z| \leq \min\{1, \frac{|\alpha_d|}{2} e^{-2d(d+1)^2}\}$ , we have

$$|\Delta_d(z)| \geq \frac{1}{2} |\alpha_d| |z|^{\eta_d}.$$

In particular in case  $|\alpha_d| \leq 2e^{2d(d+1)^2}$ , for  $z$  with  $|z| = \frac{|\alpha_d|}{2} e^{-2d(d+1)^2}$ , we have

$$|\Delta_d(z)| \geq \left| \frac{\alpha_d}{2} \right|^{\eta_d+1} e^{-2d(d+1)^2 \eta_d},$$

and in case  $|\alpha_d| \geq 2e^{2d(d+1)^2}$ , we have for any  $z$  with  $|z| = a \leq 1$

$$|\Delta_d(z)| \geq e^{2d(d+1)^2} a^{\eta_d}.$$

*Proof.* Since  $|f|$  is bounded by 1 on  $D_2$ , it is also the case for its powers  $f^j$ ,  $j \geq 1$ . By Cauchy's estimates, we thus have  $|a_i^j(z)| \leq 1$ , for any  $z \in D_1$ . Therefore the Bautin determinant  $\Delta_d(z)$  does not exceed on  $D_1$ , in absolute value,

$$(d^2 + d)! \leq e^{(d^2+d) \log(d^2+d)} \leq e^{(d^2+d) \log((d+1)^2)} \leq e^{2d(d+1)^2} = \kappa.$$

Now write  $\Delta_d(z)$  as  $\Delta_d(z) = z^{\eta_d} (\alpha_d + \sum_{k=1}^{\infty} \gamma_k z^k)$ . Again by Cauchy's estimates we have

$$|\gamma_k| \leq \frac{\kappa}{2^{k+\eta_d}}, \text{ and hence, for } |z| \leq 1,$$

$$\left| \sum_{k=1}^{\infty} \gamma_k z^k \right| \leq \frac{\kappa}{2^{\eta_d}} \sum_{k=1}^{\infty} \frac{|z|^k}{2^k} \leq \kappa |z|$$

Consequently, for  $|z| \leq \min\{1, \frac{|\alpha_d|}{2\kappa}\}$  we get

$$|\Delta_d(z)| = |z|^{\eta_d} |\alpha_d| + \sum_{k=1}^{\infty} \gamma_k |z|^k \geq \frac{1}{2} |\alpha_d| |z|^{\eta_d}.$$

Finally in case  $|\alpha_d| \leq 2\kappa$ , for  $z$  with  $|z| = \frac{|\alpha_d|}{2} e^{-2d(d+1)^2}$ , we have

$$|\Delta_d(z)| \geq \left|\frac{\alpha_d}{2}\right|^{\eta_d+1} \kappa^{-\eta_d} = \left|\frac{\alpha_d}{2}\right|^{\eta_d+1} e^{-2d(d+1)^2 \eta_d},$$

and in case  $|\alpha_d| \geq 2\kappa$ , we have for any  $z$  with  $|z| \leq 1$

$$|\Delta_d(z)| \geq \frac{|\alpha_d|}{2} |z|^{\eta_d} \geq \kappa |z|^{\eta_d} = e^{2d(d+1)^2} |z|^{\eta_d}.$$

This complete the proof of the lemma.  $\square$

Now we assume in addition, as we did in section 3.2, that  $f(z) = \sum_{k=0}^{\infty} a_k z^k$  has rational Taylor coefficients  $a_k = \frac{m_k}{p_k}$ . Let, as in Notation 3.2.6, for any  $l \geq 1$ ,  $h_l = \max\{p_k; k = 1, \dots, l\}$ . Under this assumption we can bound from below  $|\alpha_d|$ , in terms of the  $d$ -th Bautin multiplicity  $\eta_d$  and the height bound  $h_{d^2+d+\eta_d}$ :

**Lemma 3.3.5.** *For  $f$  as above we have*

$$|\alpha_d| \geq \left[ h_{d^2+d+\eta_d}^{d(d^2+d+\eta_d)} (d^2+d)! \right]^{-(d^2+d)}$$

*Proof.* Consider for each  $j = 1, \dots, d$ , and  $p = 1, \dots, d^2+d$  the functions

$$a_p^j(z) = \frac{(f^j)^{(p)}(z)}{p!} = \sum_{k=0}^{\infty} b_k^{j,p} z^k,$$

which are the entries of the Bautin determinant  $\Delta_d(z)$ . We have

$$b_k^{j,p} = \left[ \frac{(f^j)^{(p)}(z)}{k!p!} \right]^{(k)} (0) = \frac{(f^j)^{(p+k)}(0)}{k!p!} = (p+k)! \frac{a_{p+k}^j}{k!p!} = A \frac{a_{p+k}^j}{p!}, \quad (*)$$

for a certain integer  $A$ . On the other hand

$$a_{p+k}^j = \sum_{i_1+\dots+i_j=k+p} a_{i_1}^1 \cdots a_{i_j}^1, \quad (**)$$

Now if we write the  $a_{i_l}^1$  for  $i_l = 1, \dots, k+p$ , with  $p = 1, \dots, d^2+d$  and  $k = 0, \dots, \eta_d$  as rational numbers having the same denominator  $D$ , this denominator may be bounded by  $h_{d^2+d+\eta_d}^{d^2+d+\eta_d}$ . By (\*\*) the  $a_{p+k}^j$  may be written as rational numbers having the same denominator bounded by  $D^d$ . And by (\*) the  $b_k^{j,p}$  may also be written as rational numbers having the same denominator bounded by  $D^d(d^2+d)!$ .

The coefficient  $\alpha_d$  of  $z^{\eta_d}$  in the Taylor expansion of  $\Delta_d(z)$  is expressed as sum of products of  $d^2+d$  factors of type  $b_k^{j,p}$ , with  $j = 1, \dots, d$ ,  $p = 1, \dots, d^2+d$  and



$k = 0, \dots, \eta_d$ . It follows that  $\alpha_d$  is a non zero rational number that can be written with a denominator bounded by  $(D^d(d^2 + d)!)^{d^2+d}$ . Therefore

$$|\alpha_d| \geq (D^d(d^2 + d)!)^{-(d^2+d)} \geq \left[ h_{d^2+d+\eta_d}^{d(d^2+d+\eta_d)} (d^2 + d)! \right]^{-(d^2+d)}.$$

□

Combining Lemma 3.3.4 and Lemma 3.3.5 we can now produce a lower bound for the Bautin determinant  $\Delta_d(z)$ , at some points in  $D_1$ , in terms of the Bautin multiplicity sequence  $(\eta_d)_{d \geq 1}$  and the height sequence  $(h_d)_{d \geq 1}$ . In this goal, let us assume that there exist polynomials  $R$  and  $S$ , with positive coefficients, such that

$$(3.3.2) \quad \eta_d \leq R(d), \quad h_l \leq e^{S(l)}, \quad \text{for } d, l \geq 1$$

Under this condition we show that  $\mathcal{Z}_d(f)$ , in the disk  $D_1$ , grows at most polynomially in  $d$ .

**Theorem 3.3.6.** *Assume that  $f(z) = \sum_{k=0}^{\infty} a_k z^k$  has rational Taylor coefficients, and the growth condition (3.3.2) is satisfied. Then, on  $D_1$ ,  $\mathcal{Z}_d(f)$  is bounded by  $T(d)$ , for a certain polynomial  $T$ .*

*Proof.* By Lemma 3.3.5 under condition 3.3.2 we obtain the following lower bound for the coefficient  $\alpha_d$  defined in equation (3.3.1)

$$\begin{aligned} |\alpha_d| &\geq \left[ h_{d^2+d+\eta_d}^{d(d^2+d+\eta_d)} (d^2 + d)! \right]^{-(d^2+d)} \\ &\geq e^{-d(d^2+d+R(d))(d^2+d)S(d^2+d+R(d))} e^{-(d^2+d)^3} = e^{-U(d)}, \end{aligned}$$

for some polynomial  $U$  with positive coefficients. For a given integer  $d$ , by Lemma 3.3.4, in case  $|\alpha_d| \geq 2e^{2d(d+1)^2}$ , one can find a point  $z_0 \in D_1$  such that  $|\Delta_d(z_0)| \geq e^{2d(d+1)^2} a^{\eta_d}$ , with  $|z_0| = a$  any real in  $]0, 1[$ . In particular for  $a = e^{-1}$ , one has in case  $|\alpha_d| \geq 2e^{2d(d+1)^2}$ , the existence of a points  $z_0 \in D_1$  such that

$$|\Delta_d(z_0)| \geq e^{-\eta_d} \geq e^{-R(d)}. \quad (*)$$

Still by Lemma 3.3.4, in case  $|\alpha_d| \leq 2e^{2d(d+1)^2}$ , one can find a point  $z_0 \in D_1$  such that if  $\frac{|\alpha_d|}{2} \leq 1$

$$\begin{aligned} |\Delta_d(z_0)| &\geq \left| \frac{\alpha_d}{2} \right|^{\eta_d+1} e^{-2d(d+1)^2 \eta_d} \\ &\geq e^{-(R(d)+1)(U(d)+1)} e^{-2d(d+1)^2 R(d)} = e^{-V(d)}, \end{aligned} \quad (**).$$

for some polynomial  $V$  with positive coefficients, and such that if  $\frac{|\alpha_d|}{2} \geq 1$

$$|\Delta_d(z_0)| \geq e^{-2d(d+1)^2 \eta_d} \geq e^{-2d(d+1)^2 R(d)} = e^{-W(d)}, \quad (***)$$

for some polynomial  $W$  with positive coefficients.

In any cases (\*), (\*\*) or (\*\*\*) one can find a point  $z_0$  in  $D_1$  such that  $|\Delta_d(z_0)| \geq e^{-Q(d)}$  for some polynomial  $Q$  with positive coefficients, not depending on  $z_0$ .

Now we apply Proposition 2.0.11 to the family  $\sum_{j=0}^d p_j f^j$  on the concentric disks  $D(z_0, 2)$  and  $D(z_0, 7)$  centred at  $z_0$  and respectively of radius 2 and 7. Note that at  $z_0$  the Bautin index of this family is minimal, that is  $(d+1)^2 - 1 = d^2 + 2d$ , since  $\Delta_d(z_0) \neq 0$ . Note also that in this situation  $\sigma = (d+1)^2$ . We obtain that the maximal number of zeroes on  $D(z_0, 2)$  of the family is bounded by

$$\begin{aligned} & 5(d^2 + 2d) \log(4 + 2(d+1)^3 \frac{(d+1)^{(d+1)^2}}{|\Delta_d(z_0)|} 7^{-\frac{(d+1)^2}{2}(d^2+2d)}) \\ & \leq 5(d^2 + 2d) \log(4 + \frac{2}{|\Delta_d(z_0)|}) \leq 5(d^2 + 2d) \log(4 + 2e^{Q(d)}) \leq T(d), \end{aligned}$$

for some polynomial  $T$ . To finish the proof we observe that the unit disc  $D_1$  is contained in  $D(z_0, 2)$ , for any choice of  $z_0 \in D_1$ , and therefore, on  $D_1$ ,  $\mathcal{Z}_d(f)$  is also bounded by  $T(d)$ .  $\square$

**3.4. Lacunary series.** The aim of this section is to give concrete instances of series  $f$  satisfying polynomial growth condition for the transcendency sequence  $(\nu_d)_{d \geq 1}$  considered in (3.2.1), and thus having a polynomial Bézout bound. For this we focus on lacunary series for which the computation of a bound for the transcendency sequence  $(\nu_d)_{d \geq 1}$  is possible. This case has been considered in [21, Theorem 6.1], where a family of analytic functions with  $\mathcal{Z}_d(f)$  having prescribed growth is given. Our conditions, being more flexible, improve on these earlier conditions (see Remark 3.4.7).

We begin with the following remark which improves on the estimates of Propositions 3.2.7 and 3.2.8, in case of lacunary series.

*Remark 3.4.1.* Assume that the Taylor coefficients of the series  $f$  are rational numbers and denote by  $\theta_d$  the amount of those non zero coefficients among  $a_0, \dots, a_{\nu_d}$ . Then, with exactly the same proofs adapted to this notation, Propositions 3.2.7 and 3.2.8 may be formulated as follows: the absolute value  $\Delta$  of a non zero minor determinant of size  $(d^2 + d) \times (d^2 + d)$  of  $\tilde{M}$  satisfies  $\Delta \geq h_{\theta_{2d}}^{-d^2(d+1)\theta_{2d}}$ , and consequently when there exist polynomials  $R$  and  $S$  such that  $\theta_d \leq R(d)$ ,  $h_d \leq e^{S(d)}$ , one gets  $\mathcal{Z}_d(f) \leq T(d)\nu_{2d} \log \nu_{2d}$ , for some polynomial  $T$ .

Now in order to estimate the growth of the transcendency sequence  $(\nu_d)_{d \geq 1}$ , let us assume that the lacunarity of the series  $f$  is quantitatively controlled by the following condition

$$(3.4.1) \quad f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}, \quad a_k \neq 0 \quad \text{and for any } k \geq 1, \quad n_{k+1} > n_k^2.$$

Note that in what follows no assumption is made on the rationality of the Taylor coefficients of  $f$ . We assume that  $f$  is analytic on  $D_1$  and bounded there by 1.

**Lemma 3.4.2.** *Under condition (3.4.1), for any  $l \geq 1$  and for any  $m, j \in [0, n_{l+1} - 1]$ , the series  $z^m f^j(z)$  contains the non-zero monomial  $(a_{l+1})^j z^{jn_{l+1}+m}$ , and no other monomials with degree in  $]jn_{l+1} + m, n_{l+2}[$ .*

*Proof.* For  $j = 0$  the series  $z^m f^j(z)$  is the monomial  $z^m$  and thus is of the required form. Now for  $j \geq 1$ , write  $f(z)$  as the sum  $f(z) = f_{l+1}(z) + \bar{f}_{l+1}(z)$ , with  $f_{l+1}(z) = \sum_{k=1}^{l+1} a_k z^{n_k}$ , and  $\bar{f}_{l+1}(z) = \sum_{k=l+2}^{\infty} a_k z^{n_k}$ . The monomial  $(a_{l+1})^j z^{jn_{l+1}}$  has the highest degree among the monomials of the series  $f^j(z)$  which come from the terms in  $f_{l+1}^j$ . On the other hand, all other monomials in  $f^j(z)$  which come from the products of terms in  $f_{l+1}(z)$  and in  $\bar{f}_{l+1}(z)$ , have degree at least  $n_{l+2}$ . The terms of the series  $x^m f^j(z)$  are the terms of  $f^j(z)$  shifted by  $m$  and therefore for  $m, j < n_{l+1}$  we have

$$jn_{l+1} + m \leq (n_{l+1} - 1)n_{l+1} + n_{l+1} - 1 = n_{l+1}^2 - 1 < n_{l+2},$$

since by condition (3.4.1),  $n_{l+2} > n_{l+1}^2$ .  $\square$

**Proposition 3.4.3.** *Under condition (3.4.1), for any  $l \geq 1$  and for any  $d$  in the interval  $[n_l, n_{l+1} - 1]$  we have*

$$n_{l+1} \leq \nu_d \leq n_{l+1}^2 - 1 < n_{l+2}.$$

*Proof.* First of all, for  $l \geq 1$  and  $d \geq n_l$ , we have  $\nu_d \geq \nu_{n_l} \geq n_{l+1}$ , since for the polynomial  $P(z, y) = y - \sum_{k=1}^l a_k z^{n_k}$  of degree  $n_l$ , the function  $P(z, f(z)) = \sum_{k=l+1}^{\infty} a_k z^{n_k}$

has multiplicity  $n_{l+1}$  at the origin.

Let now  $P_d(z, y) = p_d(z)y^d + \dots + p_1(z)y + p_0(z)$  be a polynomial of degree  $d \leq n_{l+1} - 1$  and let us prove that the multiplicity of  $P_d(z, f(z))$  at the origin is at most  $n_{l+1}^2 - 1$ ; this will prove that  $\nu_d \leq n_{l+1}^2 - 1 < n_{l+2}$ .

Denote by  $s \leq d$  the highest degree of  $y$  in  $P_d(z, y)$  for which the polynomial  $p_s(z)$  is not identically zero, and let us write  $p_s(z) = ax^r + bx^{r-1} + \dots$ ,  $r \leq d$ , with  $a \neq 0$ . By Lemma 3.4.2, the summand  $ax^r f^s(z)$  in  $P_d(z, f(z))$  contains the monomial  $v = a(a_{l+1})^s z^{sn_{l+1}+r}$ . Let us show that this monomial cannot cancel with any other monomial in  $P_d(z, f(z))$ , because, if it is the case, since for  $s, r \leq d < n_{l+1}$ , we have

$$sn_{l+1} + r \leq (n_{l+1} - 1)n_{l+1} + n_{l+1} - 1 \leq n_{l+1}^2 - 1 < n_{l+2},$$

it will finish the proof. As just noticed, since  $sn_{l+1} + r < n_{l+2}$ , the monomial  $v$  can

cancel only with the monomials coming from the truncated series  $f_{l+1}(z) = \sum_{k=1}^{l+1} a_k z^{n_k}$

introduced in the proof of Lemma 3.4.2, since  $f(z) - f_{l+1}(z) = \sum_{k=l+2}^{\infty} a_k z^{n_k}$ . But on one

hand  $v$  cannot cancel with any monomial in  $p_s(z)f_{l+1}^s(z)$ , and on the other hand, for any  $q < s$ , the monomials in  $f^q(z)$ ,  $q < s$ , coming from the truncated series  $f_{l+1}(z)$ , have degree at most  $qn_{l+1}$ . Hence the highest degree of monomials in  $p_q(z)f_{l+1}^q(z)$  can be  $qn_{l+1} + d < (q+1)n_{l+1} \leq sn_{l+1} + r$ . As announced, we conclude that  $v$  cannot cancel with any other monomial in  $P_d(z, f(z))$ .  $\square$

Lemma 3.4.2 does not only let us bound the terms of the sequence  $\nu$  like in Proposition 3.4.3, it also lets us compute some  $(d+1)^2 \times (d+1)^2$  non zero minor determinant in the Bautin matrix  $M$  (up to allowing more than  $b+1$  rows in  $M$ ).

**Proposition 3.4.4.** *Under condition (3.4.1), for any  $l \geq 1$ , for any  $d \in [n_l, n_{l+1} - 1]$ , there exists a  $(d+1)^2 \times (d+1)^2$  minor in  $M$  (up to allowing more than  $b+1$  rows in the definition of the Bautin matrix  $M$ ) with non zero determinant  $\Delta = (a_{l+1})^{\frac{1}{2}d(d+1)^2}$ . For  $d = n_{l+1} - 1$  this determinant is the upper square  $(d+1)^2 \times (d+1)^2$  minor of the Bautin matrix  $M$ .*

*Proof.* Let us fix  $l \geq 1$ , and  $d \in [n_l, n_{l+1} - 1]$ . Then for any  $j = 0, \dots, d$  by Lemma 3.4.2, we have in  $M$  a lower-triangular square  $(d+1) \times (d+1)$  block  $M_j$  corresponding to lines ranging from  $jn_{l+1}$  to  $jn_{l+1} + d$  and columns ranging from  $j(d+1)$  to  $(j+1)(d+1) - 1$  in  $M$ , and  $M_j$  has for entries  $(a_{l+1})^j$  on its main diagonal. Note that  $dn_{l+1} + d$  may be bigger than  $b+1$ , so we maybe have to consider a matrix having more lines than the Bautin matrix  $M$ , which is harmless.

Now, if we drop from  $M$  all the lines which are not in  $M_j$  for some  $j = 0, \dots, d$ , we obtain a  $(d+1)^2 \times (d+1)^2$  minor  $M'$  which is lower triangular and has the blocks  $M_j$ ,  $j = 0, \dots, d$ , on its main diagonal. Hence its determinant  $\Delta$  is  $(a_{l+1})^{\frac{1}{2}d(d+1)^2}$ .

For  $d$  equal to its maximal value  $n_{l+1} - 1$  each line of  $M$  belongs to one block  $M_j$ , and hence  $M'$  coincides with the upper square  $(d+1)^2 \times (d+1)^2$  minor of the Bautin matrix  $M$ .  $\square$

So far, in condition (3.4.1) we have required that the lacunarity of the series  $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$  is big enough. We now require in addition that the lacunarity of  $f$  is not too big, in the following condition

$$(3.4.2) \quad \text{There exists } q > 2, \text{ such that for any } k \geq 1, n_k^2 < n_{k+1} \leq n_k^q.$$

Under this assumption we can now show that  $f$  has a polynomial Bézout bound on  $D_{\frac{1}{4}}$ .

**Theorem 3.4.5.** *Under condition (3.4.2), the transcendency sequence  $\nu_d$  grows at most polynomially in  $d$ . More accurately we have*

$$\nu_d(f) < d^{q^2}.$$

*If, in addition, for a certain fixed  $p > 0$ , and for any  $k \geq 1$  we have  $|a_k| \geq e^{-n_k^p}$ , then on  $D_{\frac{1}{4}}$*

$$\mathcal{Z}_d(f) \leq 10(2d)^{q^2}(1 + qd^2 + 5d^{pq+3}).$$

*Proof.* Let the degree  $d \geq 1$  be given and let  $l$  be such that  $d \in [n_l, n_{l+1} - 1]$ . By Proposition 3.4.3 we have

$$\nu_d < n_{l+2} \leq (n_l)^{q^2} \leq d^{q^2}.$$

Now by Proposition 3.4.4 we obtain the existence of some  $(d+1)^2 \times (d+1)^2$  minor in  $M$  with non zero determinant  $\Delta$ , such that

$$\begin{aligned} |\Delta| &= |a_{l+1}|^{\frac{1}{2}d(d+1)^2} \geq \exp\left(-\frac{1}{2}d(d+1)^2 n_{l+1}^p\right) \\ &\geq \exp\left(-\frac{1}{2}d(d+1)^2 n_l^{qp}\right) \geq \exp\left(-\frac{1}{2}d^{qp+1}(d+1)^2\right). \end{aligned}$$

Now by Proposition 3.2.5,

$$\begin{aligned} \mathcal{Z}_d(f) &\leq 5\nu_{2d} \log\left(4 + \frac{\nu_{2d} + 1}{\Delta} e^{2(d+1)^3}\right) \leq \\ &\leq (2d)^{q^2} 5 \log\left(4 + ((2d)^{q^2} + 1) e^{\frac{1}{2}d^{qp+1}(d+1)^2 + 2(d+1)^3}\right). \end{aligned}$$

Since for instance,  $1 + (2d)^{q^2} \leq e^{2dq^2}$ ,  $\frac{1}{2}d^{qp+1}(d+1)^2 + 2(d+1)^3 \leq 5d^{pq+1}(d+1)^2 \leq 10d^{pq+3}$ , we have

$$\mathcal{Z}_d(f) \leq (2d)^{q^2} 5 \log(4 + e^{2dq^2 + 10d^{pq+3}}).$$

And finally since  $\log(4 + e^x) \leq 2 + x$ , for  $x \geq 0$ , we obtain

$$\mathcal{Z}_d(f) \leq 10(2d)^{q^2} (1 + qd^2 + 5d^{pq+3}).$$

This completes the proof of Theorem 3.4.5  $\square$

*Remark 3.4.6.* When the gaps between the degrees  $n_l$  grow faster than assumed in Theorem 3.4.5, that is faster than forced by condition (3.4.2), the asymptotic growth of the transcendence indices  $\nu_d$  and of bounds on  $\mathcal{Z}_d(f)$  fastens accordingly. Notice, however, that from Proposition 3.4.3, for a subsequence of degrees of the form  $d = n_{l+1} - 1$ ,  $l \geq 1$ , under condition (3.4.1) only, the multiplicity  $\nu_d$  is at most  $d^2 + 2d$ . Under appropriate assumptions on the coefficients  $a_l$  for the above subsequence of degrees  $d$  we obtain a polynomial bound also for  $\mathcal{Z}_d(f)$ . This phenomenon is to compare to a similar behaviour in [22, Theorem 1.1] or [39, Theorem 0.3], [40, Theorem 1.3], where the minimal asymptotic for  $\mathcal{Z}_d(f)$  is obtained on some sequence of degrees  $d$  going to infinity.

*Remark 3.4.7.* In [21, Corollary 6.2], for  $\alpha \geq 3$ , the lacunarity condition,

$$(3.4.3) \quad n_{k+1} = n_k^{\alpha-1}, \quad a_k = e^{-n_k \log n_k}$$

is considered as a condition implying a polynomial Bézout bound for  $f$  (with prescribed order of asymptotic in  $[\alpha - 1, \alpha]$  as explained in the introduction). Here our condition (3.4.2) and our assumption  $|a_k| \geq e^{-n_k^p}$ , for some  $p > 0$ , in Theorem 3.4.5, are somewhat more flexible as conditions giving a polynomial Bézout bound for  $f$ .

**3.5. Rational Taylor coefficients via recurrence relations.** In this section we

study recurrence relations for the Taylor coefficients of the series  $f(z) = \sum_{k=1}^{\infty} a_k z^k$ ,

assuming that the starting coefficients are rational numbers. In case the recurrence relation is linear (with polynomial coefficients in  $1/k$ ) it turns out that the  $a_k$ 's satisfy the bound  $h_l \leq e^{S(l)}$ ,  $l \geq 1$  of conditions (3.3.2) and (3.2.1), one of the two hypotheses required in Theorems 3.2.8 and 3.3.6. Furthermore in case  $f$  is a solution of a linear differential equation with polynomial coefficients, the other hypothesis required in Theorem 3.2.8, namely a bound that is polynomial in  $d$  for the transcendence index  $\nu_d$ , is automatically satisfied (see Remark 3.5.2 and Theorem 3.5.3).

Let

$$Q(k, u_1, u_2, \dots, u_r) = \sum_{|\beta| \leq d_1} p_\beta(k) u^\beta$$

be a polynomial of degree  $d_1$  in the variables  $u_1, u_2, \dots, u_r$  and with coefficients  $p_\beta(k) = \sum_{i=0}^{d_2} c_{\beta,i} (\frac{1}{k})^i$  being polynomials in  $\frac{1}{k}$  of degree  $d_2$ . We consider a polynomial recurrence relation of length  $r$  of the form

$$(3.5.1) \quad a_{k+1} = Q(k, a_k, a_{k-1}, \dots, a_{k-r+1}).$$

We assume that the coefficients  $c_{\beta,i}$  are rational numbers, as well as the initial terms  $a_0, a_1, \dots, a_{r-1}$  of the sequence  $a = (a_0, a_1, \dots, a_{r-1}, a_r, \dots)$ . For  $k \geq r-1$ , let  $D_k$  denote the common denominator of  $a_0, \dots, a_k$ , when those rational numbers are written in their irreducible form. We also denote by  $L_1$  (respectively,  $L_2$ ) the common denominator of all the coefficients  $c_{\beta,i}$ ,  $i = 0, \dots, d_2$ ,  $|\beta| \leq d_1$  (respectively, the common denominator of all the initial given terms  $a_0, a_1, \dots, a_{r-1}$ ) again when those rational numbers are written in their irreducible form. Note that  $D_{r-1} = L_2$ .

**Proposition 3.5.1.** *With the notation above, for any  $k \geq r-1$ ,*

$$D_k \leq e^{M d_1^{k-r+1} k \log k},$$

where

$$M = \max \left( \frac{\log L_2}{(r-1) \log(r-1)}, d_2 + \frac{\log L_1}{\log 2} \right).$$

*Proof.* The products of  $a_j$  entering  $Q$  in (3.5.1) can be written with denominator  $D_k^{d_1}$ . Therefore, the next term  $a_{k+1}$ , and hence all the terms  $a_0, \dots, a_k, a_{k+1}$ , can be written with the common denominator  $\tilde{D}_{k+1} = L_1 k^{d_2} D_k^{d_1}$ . Now we prove by induction that

$$(3.5.2) \quad D_k \leq e^{M d_1^{k-r+1} k \log k}.$$

For  $k = r-1$  we have  $D_{r-1} = L_2$ , and (3.5.2) is satisfied by the choice of  $M$ . Assuming that the required inequality is satisfied for a certain  $k \geq r-1$ , we now prove it for  $k+1$ . We have

$$\begin{aligned} D_{k+1} &\leq \tilde{D}_{k+1} = L_1 k^{d_2} D_k^{d_1} \leq L_1 k^{d_2} e^{M d_1^{k+1-r+1} k \log k} \\ &= e^{M d_1^{k+1-r+1} k \log k + \log L_1 + d_2 \log k} \\ &= e^{M d_1^{k+1-r+1} (k \log k + \frac{\log L_1 + d_2 \log k}{M d_1^{k+1-r+1}})}. \end{aligned}$$

By the choice of  $M$  the last expression does not exceed

$$e^{M d_1^{k+1-r+1} (k \log k + \log k)} < e^{M d_1^{k+1-r+1} (k+1) \log(k+1)}.$$

This completes the proof of Proposition 3.5.1.  $\square$

Notice that for  $d_1 > 1$  the denominators grow as a double exponent, i.e. faster than an exponent of a polynomial in  $k$ . The trivial example of recurrence relation  $a_{k+1} = a_k^2$ , i.e.  $a_k = a_0^{2^k}$ , shows that this growth indeed happens in recurrence relations of the form (3.5.1).

*Remark 3.5.2.* However, in the special case of linear recurrence relations of the form (3.5.1), we have  $d_1 = 1$ , and the bound of Proposition 3.5.1 takes the form  $D_k \leq e^{M k \log k}$ . This special case includes Poincaré-type recurrence relations, which are satisfied by the Taylor coefficients of solutions  $f(z)$  of linear differential equations with polynomial coefficients. Therefore, combining well-known bounds on the

transcendence sequences  $(\nu_d)_{d \geq 1}$  of solutions of linear differential equations with polynomial coefficients, that turn out to be polynomially bounded in  $d$  (see [6, 25, 32, 31]), our Proposition 3.5.1 and Theorem 3.2.8 above, we immediately obtain the following statement, a result, which was recently proved (among others results in this direction and by others methods) in [7, Corollary 4, Theorem 6].

**Theorem 3.5.3.** *Let  $f$  be a solution of a linear differential equation with coefficients in  $\mathbb{Q}[z]$  and rational initial conditions, defined on the unit disc. Then there exists a polynomial  $T$  such that on  $D_{\frac{1}{4}}$ ,  $\mathcal{Z}_d(f) \leq T(d)$ .*

#### 4. BAUTIN DETERMINANT FOR RANDOM SERIES

In this section, we discuss the behaviour of the Bautin determinant for random Taylor coefficients. We prove that in absolute value, the Bautin determinant is bounded from below by  $e^{U_p(d)}$ , for a certain polynomial  $U_p$  and for a choice of Taylor coefficients of  $f$  with probability  $p \in ]0, 1[$ . In case  $\Delta_d \neq 0$ , the Bautin index of the family  $\sum_{j=0}^d p_j f^j$  is  $d^2 + 2d$ , and consequently the transcendence index  $\nu_d$  is bounded by  $d^2 + 2d$ , and thus the growth condition on the transcendence indices sequence in Theorem 3.3.6 is fulfilled. It follows that for any  $p \in ]0, 1[$ , for Taylor coefficients in a set of probability  $p$ ,  $\mathcal{Z}_d(f)$  is polynomially bounded in  $d$ .

Let us fix some integer  $d \geq 1$  and let us start by the following remark.

*Remark 4.0.4.* As a polynomial in the Taylor coefficients of the series  $f$ , the Bautin determinant, still denoted  $\Delta_d$ , of size  $d^2 + d$  is a polynomial in the variables  $a_1^1, \dots, a_{d^2+d}^1$  and with degree  $\frac{d(d+1)^2}{2}$ . Indeed, the entries of  $\tilde{M}$  are the coefficients  $a_i^j$ , with  $i = 1, \dots, d^2 + d$ ,  $j = 1, \dots, d$ , and  $a_i^j$  is a sum of products of type  $a_{i_1}^1 \cdots a_{i_j}^1$ ,  $i_1 + \dots + i_j = i$ .

For  $d \geq 1$ , following this remark, and to be more general, we will consider instead of the polynomial  $\Delta_d$  of arity  $d^2 + d$  and degree  $d(d+1)^2/2$ , any polynomial with arity and degree polynomially bounded in  $d$ .

Let  $I = [-1, 1] \subset \mathbb{R}$ . We consider the unit  $n$ -dimensional cubes  $I^n \in \mathbb{R}^n$ ,  $n \geq 1$ , and the infinite dimensional unit cube  $I^\infty = \varprojlim_{n \in \mathbb{N}} I^n$  that comes with its standard

projections  $\pi_n : I^\infty \rightarrow I^n$ . Let us denote by  $\mu_n$  the probability Lebesgue measure on  $I^n$ , for any  $n \geq 1$ . For any  $n \geq 1$  and any measurable set  $G \subset I^n$  denote by  $\tilde{G} \subset I^\infty$  the cylinder  $\pi_n^{-1}(G)$  over  $G$ . The probability measure  $\mu$  on  $I^\infty$  is defined by setting  $\mu(E) = \sum_{i=1}^\infty \mu_{n_i}(G_{n_i})$ , for any subset  $E \subset I^\infty$  that can be expressed as a disjoint union of cylinders  $\tilde{G}_{n_i}$ , with  $G_{n_i}$  a  $\mu_{n_i}$ -measurable in  $I^{n_i}$ . Note that when  $G \subset I^i$ ,  $G' = G \times I^{j-i} \subset I^j$ , for  $1 \leq i \leq j$ , we have  $\tilde{G} = \tilde{G}'$ . But by Fubini Theorem we obtain  $\mu_i(G) = \mu_j(G')$ , since  $\mu_1(I_1) = 1$ , and thus the definition of  $\mu$  via the measures  $\mu_n$  makes sense.

We identify the sequences  $(a_k)_{k \geq 0} \in I^\infty$  with the analytic functions  $f(z) = \sum_{k=0}^\infty a_k z^k$ , this series converging at least in the interior of  $D_1$ . For a polynomial  $Q$  of arity  $m$  and for  $f = (a_k)_{k \geq 0} \in I^\infty$  we denote  $Q(a_0, \dots, a_m) = Q(\pi_m(f))$  by  $Q(f)$ .



Let finally  $(Q_d)_{d \geq 1}$  be a sequence of polynomials  $Q_d$ , of arity  $m_d$  and degree  $q_d$ , and let us assume

$$(4.0.1) \quad |Q_d|_{I^{m_d}} = \max\{|Q_d(x)|; x \in I^{m_d}\} \geq 1.$$

Note that we also have  $|\Delta_d|_{I^{d^2+d}} \geq 1$ , since  $\Delta_d = 1$  when for instance  $a_{d+1}^1 = 1$  and  $a_i^j = 0$  for  $i = 1, \dots, d^2 + d$ ,  $j = 1, \dots, d$ ,  $i \neq d + 1$ ,  $j \neq 1$ .

**Theorem 4.0.5.** *With the above notation, for any  $p \in ]0, 1[$ , there exists a set  $E_p \subset I^\infty$  of measure  $p$ , such that for any  $f \in E_p$ , and for any  $d \geq 1$ ,*

$$|Q_d(f)| \geq \left( \frac{3(1-p)}{2\pi^2 d^2 m_d} \right)^{q_d}.$$

*In particular, for  $q_d, m_d$  satisfying  $q_d \leq d^{\kappa_1}$ ,  $m_d \leq d^{\kappa_2}$ , for some  $\kappa_1, \kappa_2 \geq 0$ , we have, with probability at least  $p$ :*

$$|Q_d(f)| \geq e^{-(\gamma_p + \kappa_2)d^{\kappa_1+1}},$$

where  $\gamma_p$  goes to  $+\infty$  as  $p$  goes to 1.

*Proof.* Let  $p \in ]0, 1[$ ,  $d \geq 1$ , and  $\theta_d = \frac{6(1-p)}{\pi^2 d^2}$ . We define the real number  $\varepsilon_d$  as the maximum of the numbers  $\varepsilon$  such that the set

$$V_d = \{u \in I^{m_d}, |Q_d(u)| \leq \varepsilon\}$$

satisfies  $\mu_{m_d}(V_d) \leq \theta_d$ . Now for  $V = \bigcup_{d=0}^{\infty} V_d \subset I^\infty$  and  $E_p = I^\infty \setminus V$  we have

$$\mu(V) \leq \sum_{d=0}^{\infty} \theta_d = 1 - p, \text{ and thus } \mu(E_p) \geq p.$$

Clearly, for any  $f \in E_p$ , for any  $d \geq 1$ , we have  $|Q_d(f)| \geq \varepsilon_d$ . It remains to estimate the numbers  $\varepsilon_d$ . We use for this purpose the following multivariate Remez inequality proved in [18] (see also [14]).

*Let  $Z$  be a measurable subset of  $I^n$ . Then for every real polynomial  $P$  in  $n$  variables and of degree  $d$ ,*

$$(4.0.2) \quad |P|_{I^n} < \left( \frac{4n}{\lambda} \right)^d |P|_Z,$$

where  $\lambda = \mu_n(Z)$ .

Applying inequality (4.0.2) to  $Z = V_d$  and  $P = Q_d$ , by our assumption (4.0.1) we get

$$1 \leq |Q_d|_{I^{q_d}} < \left( \frac{4m_d}{\mu_{m_d}(V_d)} \right)^{q_d} \varepsilon_d,$$

or equivalently  $\mu_{m_d}(V_d) \leq 4m_d \varepsilon_d^{\frac{1}{q_d}}$ . In particular, for  $\varepsilon_d = \left( \frac{\theta_d}{4m_d} \right)^{q_d} = \left( \frac{3(1-p)}{2\pi^2 d^2 m_d} \right)^{q_d}$ , we have  $\mu_{m_d}(V_d) \leq \theta_d$ . This completes the proof of the first inequality of Theorem

4.0.5. Substituting into this inequality  $q_d = d^{\kappa_1}$ ,  $m_d = d^{\kappa_2}$ , we obtain for any  $f \in E_p$  and for any  $d \geq 1$

$$\begin{aligned} |Q_d(f)| &\geq \left( \frac{3(1-p)}{2\pi^2} \right)^{d^{\kappa_1}} d^{-2-\kappa_2 d^{\kappa_1}} = e^{-c_p d^{\kappa_1}} e^{-(2+\kappa_2 d^{\kappa_1}) \log d} \\ &\geq e^{-c_p d^{\kappa_1}} e^{-(2+\kappa_2 d^{\kappa_1}) d} \geq e^{-(2+c_p+\kappa_2) d^{\kappa_1+1}} = e^{-(\gamma_p+\kappa_2) d^{\kappa_1+1}}, \end{aligned}$$

where  $c_p = \log \left( \frac{2\pi^2}{3(1-p)} \right) > 0$ ,  $\gamma_p = 2 + c_p$ .

□

As a consequence of Theorems 4.0.5 and 3.2.8, we have the following statement.

**Corollary 4.0.6.** *With the above notation, for any  $p \in ]0, 1[$ , there exists a set  $E_p \subset I^\infty$  of measure  $p$ , such that for any  $f \in E_p$ , on  $D_{\frac{1}{4}}$ ,  $\mathcal{Z}_d(f) \leq T_p(d)$  for a certain polynomial  $T_p$ . Or, in other words, random (with respect to the measure  $\mu$ ) series satisfy polynomial Bézout bounds.*

## 5. ANALYTIC FUNCTIONS WITH FEW RATIONAL POINTS IN THEIR GRAPH

We start this section by the following definition.

**Definition 5.0.7.** Let  $x = (x_1, \dots, x_n) \in \mathbb{Q}^n$ . The height of  $x$  is the integer  $\max\{|a_i|, |b_i|; i = 1, \dots, n\}$ , where  $x_i = a_i/b_i$  with  $a_i, b_i \in \mathbb{Z}$ ,  $a_i \wedge b_i = 1$ ,  $i = 1, \dots, n$ .

Explicit bounds on the number  $\#X(\mathbb{Q}, T)$  of rational points  $X(\mathbb{Q}, T)$  of height at most  $T$ , in some given set  $X \subset \mathbb{R}^n$ , are usually related to Bézout bounds satisfied by  $X$ . Let us assume for instance that  $X$  is a transcendental set definable in some o-minimal structure expanding the real field, and to simplify, of dimension 1. Then following [36], that generalizes the classical by now method of [10], one knows that  $X(\mathbb{Q}, T)$  is contained in a certain number  $H_{X,T,d}$  of hypersurfaces of  $\mathbb{R}^n$  of degree  $d$ , this number being of the form  $C_{X,d} T^{\tau_d}$ , with  $\tau_d \rightarrow 0$  when  $d \rightarrow \infty$ . It follows that since the definable set  $X$  satisfies a Bézout bound (see [10, Theorem 1], [36])

$$(5.0.1) \quad \forall \epsilon > 0, \exists C_{X,\epsilon}, \forall T \geq 1, \#X(\mathbb{Q}, T) \leq C_{X,\epsilon} T^\epsilon.$$

Now in case the curve  $X$  is given by a system of convenient parametrizations (as the a mild parametrization defined in [35], or slow parametrizations defined in [23]), or more simply, in case  $X$  is the graph  $\Gamma_f$  of some transcendental analytic function  $f$  on a compact interval of  $\mathbb{R}$ , a computation shows that the constant  $C_{\Gamma_f,d}$  is polynomially bounded in  $d$  and that  $T^{\tau[\log T]}$  is a constant  $K$  independent of  $T$  (see [35, Proposition 2.4], [23, Proposition 2.18]). Therefore, for some polynomial  $Q$ , on gets

$$(5.0.2) \quad \#X(\mathbb{Q}, T) \leq Z_{[\log T]}(f) K Q([\log T]).$$

Moreover, in this situation when  $f$  has a Bézout bound polynomial in  $d$ , one obtains the following improvement of the general bound (5.0.1)

$$(5.0.3) \quad \exists \beta, \exists \alpha > 0, \forall T \geq 1, \#X(\mathbb{Q}, T) \leq \beta \log^\alpha T.$$

Recently several results appeared, establishing in different cases bounds for  $\#X(\mathbb{Q}, T)$  as in (5.0.3), some of them proving the existence of convenient parametrizations for certain families of sets  $X$  with respect to log-bounds as in (5.0.3), the others proving

polynomial Bézout bounds in some particular cases (see, among these results, [4], [5], [9], [7], [8], [12], [13], [19], [23], [28], [29] and [30]). In the same spirit we give hereafter direct Diophantine applications of the polynomial Bézout bounds obtained in previous sections of the paper.

Let  $f$  be an analytic function converging on  $D_1$  (on  $D_8$  for condition 3 of Theorem 5.0.8) and let us denote by  $\Gamma_f$  its graph over  $D_{\frac{1}{4}}$ . As a consequence of Theorems 3.2.8, 3.3.6, 3.4.5, 3.5.3 and Corollary 4.0.6, one has

**Theorem 5.0.8.** *Assume that one of the following conditions is satisfied*

- (1) *The Taylor coefficient of  $f$  at the origin are rational and the growth conditions (3.2.1) are satisfied,*
- (2) *The Taylor coefficient of  $f$  at the origin are rational and the growth conditions (3.3.2) are satisfied,*
- (3)  *$f(z) = \sum_{k \geq 1} a_k z^{n_k}$ , the lacunarity condition (3.4.2) is satisfied and for some  $p > 0$ , for any  $k \geq 1$ ,  $|a_k| \geq e^{-n_k^p}$ ,*
- (4)  *$f$  is a solution of a linear differential equation with polynomial coefficients in  $\mathbb{Q}[X]$  and with rational initial conditions,*
- (5)  *$f$  is a random series, in the sense of Section 4.*

*Then there exist  $\alpha, \beta > 0$  such that*

$$\#\Gamma_f(\mathbb{Q}, T) \leq \beta \log^\alpha T.$$

*Remark 5.0.9.* Not only for functions definable in some o-minimal structures, but also for analytic functions, the asymptotics of (5.0.1) is sharp, since, for instance by [34, Example 7.5], [39] or [40], there exist functions analytic on a neighbourhood of a compact interval having asymptotically as many as possible rational points of height at most  $T$  in their graph with respect to (5.0.1). For instance, for any  $\epsilon \in ]0, 1[$ , more than  $\frac{1}{2}e^{2\log^{1-\epsilon} T}$  points, for an infinite sequence of heights  $T$ . In consequence one cannot expect polynomial Bézout bounds in all degree  $d$  for these analytic functions, since by (5.0.2) one has

$$\frac{1}{2}e^{2\lceil \log T \rceil^{1-\epsilon}} \leq \frac{1}{2}e^{2\log^{1-\epsilon} T} \leq Z_{\lceil \log T \rceil}(f)C_{\Gamma_f, \lceil \log T \rceil} T^{\tau \lceil \log T \rceil}.$$

And thus for any  $\zeta \in ]0, 1[$ , there exists a sequence of degrees  $d$  going to infinity and such that

$$(5.0.4) \quad \mathcal{Z}_d(f) \geq Z_d(f) \geq e^{d^\zeta}.$$

As a consequence of (5.0.4), the condition that  $\Delta_d \geq e^{-U(d)}$  for some positive polynomial  $U$ , may not be satisfied for particular analytic functions  $f$ . Indeed, when  $\Delta_d \neq 0$ , the transcendence index  $\nu_d$  is polynomially bounded in  $d$  and in case  $\Delta_d \geq e^{-U(d)}$ , by Proposition 3.2.5

$$\begin{aligned} \mathcal{Z}_d(f) &\leq 5\nu_{2d} \log(4 + (\nu_{2d} + 1) \frac{e^{2(d+1)^3}}{\Delta}) \\ &\leq 5\nu_{2d} [\log(5) + \log(\nu_{2d} + 1) + 2(d+1)^3 + U(d)]. \end{aligned}$$

*Remark 5.0.10.* The condition  $|a_k| \geq e^{-n_k^p}$  of Theorem 5.0.8 (4) allows in particular order 0 for the lacunary series  $f(z) = \sum_{k \geq 1} a_k z^{n_k}$  when  $f$  is an entire function, since

the order of  $f$  is given by  $\limsup_{n \rightarrow \infty} -\frac{n_k \log n_k}{\log a_k}$  (see [27, Theorem 14.1.1]), contrariwise to [12, Theorem 1.1] and [22, Section 7] where order 0 is not allowed. Furthermore the conditions of Theorem 5.0.8 allow to consider analytic functions that are not entire.

*Remark 5.0.11.* The conditions 1 to 4 on  $f$  in Theorem 5.0.8 are natural with the aim of showing that there are few rational points in  $\Gamma_f$ , since combinations of finer conditions are considered in order to obtain more remarkable Diophantine properties for  $\Gamma_f$ . For instance, in Siegel-Shidlovskii's theorem a combination of conditions comparable to conditions 1 and 4 of Theorem 5.0.8, among others, imply that there is no rational point in  $\Gamma_f$ . More accurately, let  $f$  be a  $E$ -function, that is a Taylor series  $\sum_{k=0}^{\infty} a_k z^k$  with, for simplicity, rational coefficients  $a_k$  satisfying for any  $\varepsilon > 0$

$$|a_k| = O(k^{k(\varepsilon-1)}) \quad \text{and} \quad |q_k| = O(k^{\varepsilon k}),$$

where  $q_k$  is a common denominator for  $a_0, a_1, 2!a_2, \dots, k!a_k$ . Assuming moreover that  $f$  is solution of a linear differential equation of order  $n$  with coefficients in  $\mathbb{Q}[z]$ , such that  $f, f', \dots, f^{(n-1)}$  are algebraically independent over  $\mathbb{C}(z)$ , then for any algebraic number  $z_0 \neq 0$ , the numbers  $f(z_0), \dots, f^{(n-1)}(z_0)$  are algebraically independent over  $\mathbb{Q}$ , and in particular  $f(z_0)$  is transcendental (see for instance [32, Theorem 2.1] or [38, Theorem 3, p. 123]).

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